DISCRETE HILBERT TRANSFORM À LA GUNDY–VAROPOULOS

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ABSTRACT. We show that the centered discrete Hilbert transform on integers applied to a function can be written as the conditional expectation of a transform of stochastic integrals, where the stochastic processes considered have jump components. The stochastic representation of the function and that of its Hilbert transform are under differential subordination and orthogonality relation with respect to the sharp bracket of quadratic covariation. This illustrates the Cauchy Riemann relations of analytic functions in this setting. This result is inspired by the seminal work of Gundy and Varopoulos on stochastic representation of the Hilbert transform in the continuous setting.

1. INTRODUCTION

The subject of discrete analyticity and discrete Cauchy–Riemann relations originated in FERRAND [4]. The relationship of Cauchy–Riemann relations to a certain discrete Hilbert transform was understood in DUFFIN [2] together with the corresponding notions of discrete harmonic conjugate functions.

The Hilbert transform also appeared in relation with the RIESZ–TITCHMARSH transform as described in the significant discovery by MATSAEV [7]. See also MATSAEV–SODIN [8].

One should keep in mind that there are different naive ways of defining discrete Hilbert transforms from the space of sequences $\ell^2(\mathbb{Z})$ onto itself. A first manner found in the litterature is to mimic the continuous Hilbert transform $\mathcal{H}_{\mathbb{R}}$ defined on the real line as

$$\forall x \in \mathbb{R}, \quad \mathcal{H}_{\mathbb{R}}(f)(x) = -\frac{1}{\pi}\int \frac{f(x-y)}{y}\mathrm{d}y,$$

where the integral is to be understood in the principal value sense. Indeed, a naive discrete counterpart of $\mathcal{H}_{\mathbb{R}}$ defined thanks to the discrete convolution

$$\mathcal{H}_{\mathbb{Z}} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \quad \forall x \in \mathbb{Z}, \quad \mathcal{H}_{\mathbb{Z}}(f)(x) = -\frac{1}{\pi}\sum_{z\in\mathbb{Z}_*} \frac{f(x-z)}{z}$$

preserves the idea of a principal value integral by skipping $z = 0$ in the summation, but lacks many other important features of the continuous Hilbert transform. For instance, the operator $\mathcal{H}_{\mathbb{R}}$ is an isometry in $L^2(\mathbb{R})$, an anti-involution, i.e. $\mathcal{H}_{\mathbb{R}}^2 := \mathcal{H}_{\mathbb{R}} \circ \mathcal{H}_{\mathbb{R}} = -\operatorname{Id}_{L^2(\mathbb{R})}$ and obeys $\mathcal{H}_{\mathbb{R}} \circ \sqrt{-\Delta_x} = \partial_x$. However, $\mathcal{H}_{\mathbb{Z}}$ above does not possess these basic properties. One observes that the iterate $\mathcal{H}_{\mathbb{Z}}^2 = \mathcal{H}_{\mathbb{Z}} \circ \mathcal{H}_{\mathbb{Z}}$ is far from being neither an isometry of $\ell^2(\mathbb{Z})$ nor an anti-involution. One reason for

that is the fact that the summation in the discrete convolution excludes the integer $z = 0$.

The L^p norm of any discrete Hilbert transform is a very famous open question. Optimal norm estimates are only known in the continuous case – see PICHORIDES [9] and ESSEN [3] – whose proofs have a probabilistic reinterpretation, in part through the formulae of GUNDY-VAROPOULOS [5]. Inspired by this fact, we aim at an understanding of a discrete Hilbert transform through a stochastic integral formula, resembling the continuous analog of GUNDY-VAROPOULOS.

In this paper, we take the following route, as done in LUST–PIQUARD [6]. Modelled after the defining equation for the continuous Hilbert transform, $\mathcal{H}_{\mathbb{R}} \circ \sqrt{-\Delta_x} = \partial_x$, let us recall the definition of the discrete Laplacian on $\mathbb{Z}$: we define the discrete derivatives as

$$(\partial_x^+ f)(x) := f(x+1) - f(x), \quad (\partial_x^- f)(x) := f(x) - f(x-1).$$

In $\ell^2(\mathbb{Z})$, it follows that $(\partial_x^\pm)^\star = -\partial_x^\mp$, $\Delta_x = \partial_x^+ \partial_x^- = \partial_x^- \partial_x^+ = \partial_x^+ - \partial_x^-$, and $(-\Delta_x) = (\partial_x^\pm)^\star(\partial_x^\pm)$. Equipped with these discrete operators, another classical definition for discrete Hilbert transforms is given by

$$\mathcal{H}^\pm \circ \sqrt{-\Delta_x} = \partial_x^\pm.$$

Those are the **left** (resp. **right**) discrete Hilbert transform $\mathcal{H}^+$ (resp. $\mathcal{H}^-$), most often used when defining Riesz transforms on discrete groups (see LUST–PIQUARD [6] for more details and applications to the discrete Riesz vector). Through explicit and simple calculations, we are going to see in the next section that

$$\mathcal{H}^+\mathcal{H}^- = \mathcal{H}^-\mathcal{H}^+ = -\operatorname{Id}, \qquad \|\mathcal{H}^\pm\|_{l^2(\mathbb{Z})\to l^2(\mathbb{Z})} = 1$$

and that the kernels of these operators $\mathcal{H}^\pm$ are

$$-\frac{1}{\pi}\frac{1}{n \pm 1/2}$$

respectively.

The Fourier multiplier of $\mathcal{H}_{\mathbb{R}}$ is constant on positive and negative frequencies respectively. This is a feature the operators $\mathcal{H}^\pm$ lack. In fact, these operators have Fourier multipliers that are a modulation of the square wave function.

Another important and meaningful role is played by the interplay of $\mathcal{H}_{\mathbb{R}}$ and harmonic conjugate functions. If u^f is the Poisson extension of a function f to the upper half plane and v^f the Poisson extension of $\mathcal{H}_{\mathbb{R}} f$, then the pair u^f and v^f obeys Cauchy-Riemann relations. Using space–time Brownian motion, GUNDY and VAROPOULOS have identified pairs of martingales M_t^f and $M_t^{\mathcal{H}_{\mathbb{R}} f}$ that are orthogonal and have differential subordination. In fact, in their formula, $M_t^{\mathcal{H}_{\mathbb{R}} f}$ is a martingale transform of M_t^f. The discrete counterpart of this feature of $\mathcal{H}_{\mathbb{R}}$ is the main focus of our note. Using stochastic integrals driven by semidiscrete random walks in the (semidiscrete) upper–half space, we will see that the **centered** discrete Hilbert transform defined as

$$\mathcal{H} = \frac{1}{2}(\mathcal{H}^+ + \mathcal{H}^-)$$

enjoys this stochastic representation. Indeed, we obtain for $\mathcal{H}$ certain Cauchy-Riemann relations and an analog of the GUNDY–VAROPOULOS formula. In fact, $M_t^{\mathcal{H} f}$ and M_t^f are orthogonal and differentially subordinate with respect to the sharp bracket $\langle \cdot, \cdot \rangle$.

The main goal of the paper is to prove the following representation formula *à la* GUNDY–VAROPOULOS [5]:

Theorem 1. ***(Stochastic representation)*** *The centered discrete Hilbert transform $\mathcal{H}f$ of a function $f \in L^2(\mathbb{Z})$ as defined in (??) can be written as the conditional expectation*
$$\forall x \in \mathbb{Z}, \quad \mathcal{H}f(x) = \mathbb{E}(N_0^f | \mathcal{Z}_0 = (x, 0))$$
where N_t^f, $-\infty < t \leqslant 0$, is a suitable martingale transform of a martingale M_t^f associated to f, and $\mathcal{Z}_\sqcup$ is a suitable semidiscrete random walk on the semidiscrete upper–half space $\mathbb{Z} \times \mathbb{R}^+$.

Outline of the paper. The next section is devoted to a few basic properties of the discrete hilbert transforms mentionned above. Section 3 provides semidiscrete Poisson extensions, weak formulations and semidiscrete Cauchy–Riemann relations. We introduce the relevant stochastic integrals, martingale transforms and quadratic covariations in Section 4. Finally, we prove the representation *à la* GUNDY–VAROPOULOS of the centered discrete Hilbert transform in Section 5.

2. Basic properties

Let $\mathcal{F}$ be the discrete fourier transform
$$\begin{aligned} \mathcal{F} \quad : \quad & \ell^2(\mathbb{Z}) \longrightarrow L^2\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right) \\ & \mathcal{F}(f)(\xi) := \hat{f}(\xi) = \sum_{x\in\mathbb{Z}} f(x) \quad e^{-\mathrm{i}2\pi x\xi} \end{aligned}$$
with Fourier inverse
$$\begin{aligned} \mathcal{F}^{-1} \quad : \quad & L^2\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right) \longrightarrow \ell^2(\mathbb{Z}) \\ & \mathcal{F}^{-1}(\hat{f})(x) = \int_{-1/2}^{1/2} \hat{f}(\xi) \quad e^{+\mathrm{i}2\pi x\xi} \mathrm{d}\xi \end{aligned}$$
Through explicit and simple calculations, we are going to see that

Proposition 1. ***(Equivalent definitions of $\mathcal{H}^\pm$)***
$$\widehat{\mathcal{H}^\pm}(\xi) = e^{\pm\mathrm{i}\pi\xi} \frac{\sin(\pi\xi)}{|\sin(\pi\xi)|} = e^{\mathrm{i}\pi\xi}\,\mathrm{SQ}(\pi\xi) \tag{1}$$
$$\mathcal{H}^\pm f(x) = -\frac{1}{\pi} \sum_{z\in\mathbb{Z}} \frac{f(x-z)}{z \pm \frac{1}{2}} \tag{2}$$

As a consequence, the reader may check that we also have the following properties,

Proposition 2. *****(Basic properties)*** *The discrete Hilbert transforms $\mathcal{H}^\pm$ obey the following analogs of the continuous Hilbert transform*
$$\forall f, g \in \ell^2(\mathbb{Z}), \quad (\mathcal{H}^\pm f, g)_{\ell^2(\mathbb{Z})} = -(f, \mathcal{H}^\mp g)_{\ell^2(\mathbb{Z})}, \quad i.e. (\mathcal{H}^\pm)^\star = -\mathcal{H}^\mp \tag{3}$$
$$\mathcal{H}^+\mathcal{H}^- = \mathcal{H}^-\mathcal{H}^+ = -\,\mathrm{Id} \tag{4}$$

$$\mathcal{H}^{\pm}\mathcal{H}^{\pm} = -S_{\pm 1}\,\mathrm{Id} \tag{5}$$

$$\|\mathcal{H}^{\pm}\|_{\ell^2(\mathbb{Z})\to\ell^2(\mathbb{Z})} = 1 \tag{6}$$

where S_{+1} (resp. S_{-1}) is the right (resp. left) shift operator $(S_{\pm 1}f)(x) = f(x \mp 1)$.

Proof. **(of Proposition 1)** One has succesively

$$\begin{aligned}
\widehat{\partial_x^{\pm}}(\xi) &= e^{\pm \mathrm{i}2\pi\xi} - 1 = 2\mathrm{i}e^{\pm\mathrm{i}\pi\xi}\sin(\pi\xi)\\
\widehat{\Delta_x}(\xi) &= -4\sin^2(\pi\xi)\\
\widehat{\sqrt{-\Delta_x}}(\xi) &= 2|\sin(\pi\xi)|\\
\widehat{\mathcal{H}^{\pm}}(\xi) &= \mathrm{i}e^{\pm\mathrm{i}\pi\xi}\frac{\sin(\pi\xi)}{|\sin(\pi\xi)|} =: \mathrm{i}e^{\mathrm{i}\pi\xi} \quad \mathrm{SQ}(\pi\xi).
\end{aligned}$$

Now, computing the Fourier transform of the discrete kernel, we check

$$\begin{aligned}
\sum_{y\in\mathbb{Z}} -\frac{1}{\pi}\frac{1}{y+\frac{1}{2}}e^{-2\pi\mathrm{i}y\xi} &= -\frac{2}{\pi}\sum_{y}\frac{e^{-\mathrm{i}(2y+1)\frac{\xi}{2}}}{2y+1}e^{2\pi\mathrm{i}\frac{\xi}{2}}\\
&= -\frac{2}{\pi}\mathrm{i}e^{\pi\mathrm{i}\xi}\sum_{m}\frac{\sin\left(-2\pi(2y+1)\frac{\xi}{2}\right)}{2y+1}\\
&= -\frac{4}{\pi}\mathrm{i}e^{2\pi\mathrm{i}\frac{\xi}{2}}\sum_{m\geqslant 0}\frac{\sin\left(2\pi(2y+1)\left(-\frac{\xi}{2}\right)\right)}{2y+1}\\
&= -\mathrm{i}e^{\pi\mathrm{i}\xi}\ \mathrm{SQ}\left(-\frac{\pi\xi}{2}\right)\\
&= \mathrm{i}e^{\pi\mathrm{i}\xi}\frac{\sin(\pi\xi)}{|\sin(\pi\xi)|},
\end{aligned}$$

where we used the Fourier transform of the square wave function

$$\mathrm{SQ}(t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin\left(\frac{2\pi}{T}(2n+1)t\right)}{2n+1}.$$

Similarly, the symbol of $\mathcal{H}^-$ is $\mathrm{i}e^{-\pi\mathrm{i}\xi}\frac{\sin(\pi\xi)}{|\sin(\pi\xi)|} = -\mathrm{i}e^{-\pi i\xi}\ \mathrm{SQ}\left(-\frac{\xi}{2}\right)$ □

Finally, for the centered discrete Hilbert transform, one has

$$\begin{aligned}
(\mathcal{H})\circ\mathcal{H} &= [(\mathcal{H}^+)^2 + 2(\mathcal{H}^+)(\mathcal{H}^-) + (\mathcal{H}^-)^2]/4 = [(\mathcal{H}^+)^2 + 2(\mathcal{H}^+)(\mathcal{H}^-) + (\mathcal{H}^-)^2]/4\\
&= [(\mathcal{H}^+)S^+(\mathcal{H}^-) + 2(\mathcal{H}^+)(\mathcal{H}^-) + S^-(\mathcal{H}^+)(\mathcal{H}^-)]/4\\
&= -\left(\frac{1}{4}S_- + \frac{1}{2}\,\mathrm{Id} + \frac{1}{4}S_+\right),
\end{aligned}$$

where we recognize a smoothed version of minus the identity.

3. Semi-discrete Poisson extensions and weak formulations

Semidiscrete Poisson extensions. Defining the selfadjoint operator A as the square root $A = \sqrt{-\Delta_x}$, we set $P_y = e^{-yA}$, $y \in [0, \infty[$. The Poisson extension of a function $f \in L^2(\mathbb{Z})$ is the function $f(y, x) := (P_y f)(x)$ defined on $\mathbb{R}^+ \times \mathbb{Z}$. Note that we use the same name for f and it's Poisson extension. It follows that the function $f(y, x)$ satisfies in $(0, \infty) \times \mathbb{Z}$

$$\partial_y f(x, y) = -Af(x, y), \quad \partial_y^2 f(x, y) = A^2 f(x, y) = -\Delta_x f(x, y)$$

that is $f(x, y)$ is harmonic in $(0, \infty) \times \mathbb{Z}$:

$$(\Delta_y + \Delta_x)f = 0.$$

For any f and g in $L^2(\mathbb{Z})$, we note

$$(f, g)_{L^2(\mathbb{Z})} := \sum_{x \in \mathbb{Z}} f(x)g(x)$$

the scalar product of $L^2(\mathbb{Z})$

Moreover we have on the one hand

$$\forall f, g \in l^2(\mathbb{Z}), \quad (-\Delta_x f, g) \quad := \quad ((\partial_x^\pm)^\star(\partial_x^\pm)f, g) = (\partial_x^\pm f, \partial_x^\pm g)$$

and on the other hand

$$\forall f, g \in l^2(\mathbb{Z}), \quad (-\Delta_x f, g) \quad = \quad \langle A^2 f, g\rangle = \langle Af, Ag\rangle = \langle \partial_y f, \partial_y g\rangle$$

We will collect below all the derivations in the 4–vector

$$\nabla_{y,x} = (\partial_y, \partial_x^+, \partial_y, \partial_x^-)^\star.$$

Notice that we repeat twice the derivation ∂_y. The reasons for that will become clear later.

Theorem 2. ***(Weak formulation of the identity operator)*** *Assume f and g in $L^2(\mathbb{Z})$. Let $\mathcal{I}$ denote the identity operator. We have the Littlewood–Paley identity*

$$(\mathcal{I}f, g) = \int_{y=0}^{\infty} \sum_{x \in \mathbb{Z}} (\nabla_{x,y} f(x, y), \nabla_{x,y} g(x, y)) y \mathrm{d}y$$

Proof. Notice first that for any functions f and g in $L^2(\mathbb{Z})$, we have, using successively discrete integration by parts, and the definition of A,

$$\begin{aligned} (-\Delta_x f, g) &:= ((\partial_x^\pm)^\star(\partial_x^\pm)f, g) = (\partial_x^\pm f, \partial_x^\pm g) \\ &= (A^2 f, g) = (Af, Ag) \end{aligned}$$

In the particular case where both $f = f(x, y)$ and $g = g(x, y)$ are Poisson extensions, then

$$(Af, Ag) = (\partial_y f, \partial_y g)$$

To summarize, when $f = f(x, y)$ and $g = g(x, y)$ are Poisson extensions, we have in the upper half space

$$(-\Delta_x f, g) = (\partial_x^+ f, \partial_x^+ g) = (\partial_x^- f, \partial_x^- g) = (\partial_y f, \partial_y g) = (Af, Ag) = (\Delta_y f, g)$$

Now, for any function $F(y)$ smooth enough and decaying at infinity, we have

$$F(0) = \int_0^{\infty} F''(y) y \mathrm{d}y$$

Applying the identity above to $F(y) = (f(y), g(y))_{L^2(\mathbb{Z})}$ yields,

$$\begin{aligned}
(f,g) &= \int_0^\infty \{(\partial_y^2 f, g) + 2(\partial_y f, \partial_y g) + (f, \partial_y^2 g)\} y \mathrm{d}y \\
&= 4\int_0^\infty (\partial_y f, \partial_y g) y \mathrm{d}y \\
&= 4\int_0^\infty (\partial_x^+ f, \partial_x^+ g) y \mathrm{d}y = 4\int_0^\infty (\partial_x^- f, \partial_x^- g) y \mathrm{d}y \\
&= \int_0^\infty \{(\partial_y f, \partial_y g) + (\partial_x^+ f, \partial_x^+ g) + (\partial_y f, \partial_y g) + (\partial_x^- f, \partial_x^- g)\} y \mathrm{d}y \\
&= \int_0^\infty (\nabla_{y,x} f, \nabla_{y,x} g) y \mathrm{d}y,
\end{aligned}$$

yielding the result announced. $\square$

Cauchy–Riemman relations and weak formulation for the discrete Hilbert.

Theorem 3. ***(Cauchy-Riemann relations)*** *Let f and g in $\ell^2(\mathbb{Z})$. Let $\mathcal{H}f(y,x)$ denote the Poisson extension of $\mathcal{H}f(x)$, and $f(y,x)$ that of $f(x)$. We have the semidiscrete Cauchy–Riemman relations, for all $(y,x) \in R^+ \times \mathbb{Z}$,*

$$\partial_y \mathcal{H}^\pm f = -\partial_x^\mp f, \quad \partial_x^\pm \mathcal{H}^\pm f = \partial_y f,$$

which implies for the centered discrete Hilbert transform, for all $(y,x) \in R^+ \times \mathbb{Z}$,

$$\partial_y \mathcal{H} f = -\partial_x^0 f, \quad \partial_x^0 \mathcal{H} f = \left(\frac{1}{4} S_- + \frac{1}{2}\,\mathrm{Id} + \frac{1}{4} S_+\right) \partial_y f.$$

Theorem 4. ***(Weak formulation for the discrete Hilbert)*** *Let $\mathcal{H}f$ denote the centered discrete Hilbert transform of f. Let $f := f(y,x)$ denote the Poisson extension of f. Let $g := g(y,x)$ denote the Poisson extension of a test function g. We have the weak formulation:*

$$(\mathcal{H}f, g) = \int_0^\infty (A \nabla_{y,x} f, \nabla_{y,x} g) y \mathrm{d}y$$

where $A \in \mathbb{R}^{4\times 4}$ is the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is important to observe that the weak formulation involves the orthogonal matrix A such that $A^2 = -\mathrm{Id}$. This does not mean that the centered discrete Hilbert transform $\mathcal{H}$ is an antiinvolution, as is clear from Theorem 3 and equation (7).

Proof. The semidiscrete Poisson extension convenient for us has up to normalization the kernel

$$P_y(x) = \int_0^1 e^{-2\pi \mathrm{i} \xi x} e^{-2|\sin(\pi\xi)| y} d\xi$$

Let us write for convenience u^f the Poisson extension of f

$$u(x,y) = \int_0^1 \hat{f}(\xi) e^{-2\pi \mathrm{i} \xi x} e^{-2|\sin(\pi\xi)| y} d\xi$$

and the Poisson extension of $\mathcal{H}^{\pm}f$

$$v^{\pm}(x,y) = \int_0^1 \widehat{\mathcal{H}_{\pm}f}(\xi)e^{-2\pi \mathrm{i}\xi x}e^{-2|\sin(\pi\xi)|y}d\xi = -\mathrm{i}\int_0^1 e^{\pm\pi\mathrm{i}\xi}\frac{\sin(\pi\xi)}{|\sin(\pi\xi)|}\hat{f}(\xi)e^{-2\pi\mathrm{i}\xi x}e^{-2|\sin(\pi\xi)|y}d\xi$$

Observe that

$$\begin{aligned}\partial_x^+ u(x,y) &= \int_0^1 \hat{f}(\xi)(e^{-2\pi\mathrm{i}\xi(x+1)} - e^{-2\pi\mathrm{i}\xi x})e^{-2|\sin(\pi\xi)|y}d\xi \\ &= -2\mathrm{i}\int_0^1 e^{-\pi\mathrm{i}\xi}\sin(\pi\xi)\hat{f}(\xi)e^{-2\pi\mathrm{i}\xi x}e^{-2|\sin(\pi\xi)|y}d\xi\end{aligned}$$

While

$$\begin{aligned}\partial_x^- u(x,y) &= \int_0^1 \hat{f}(\xi)(e^{-2\pi\mathrm{i}\xi x} - e^{-2\pi\mathrm{i}\xi(x-1)})e^{-2|\sin(\pi\xi)|y}d\xi \\ &= -2\mathrm{i}\int_0^1 e^{\pi\mathrm{i}\xi}\sin(\pi\xi)\hat{f}(\xi)e^{-2\pi\mathrm{i}\xi x}e^{-2|\sin(\pi\xi)|y}d\xi\end{aligned}$$

Similarily

$$\partial_x^+ v^+(x,y) = -2\int_0^1 |\sin(\pi\xi)|\hat{f}(\xi)e^{-2\pi\mathrm{i}\xi x}e^{-2|\sin(\pi\xi)|y}d\xi$$

and

$$\partial_x^- v^-(x,y) = -2\int_0^1 |\sin(\pi\xi)|\hat{f}(\xi)e^{-2\pi\mathrm{i}\xi x}e^{-2|\sin(\pi\xi)|y}d\xi$$

The continuous derivatives in the other variable are

$$\partial_y u(x,y) = -2\int_0^1 |\sin(\pi\xi)|\hat{f}(\xi)e^{-2\pi\mathrm{i}\xi x}e^{-2|\sin(\pi\xi)|y}d\xi$$

and

$$\partial_y v^{\pm}(x,y) = 2\mathrm{i}\int_0^1 e^{\pm\pi\mathrm{i}\xi}\sin(\pi\xi)\hat{f}(\xi)e^{-2\pi\mathrm{i}\xi x}e^{-2|\sin(\pi\xi)|y}d\xi$$

This gives the following Cauchy Riemann equations

$$\partial_y v^+ = -\partial_x^- u,$$

$$\partial_y v^- = -\partial_x^+ u$$

but

$$\partial_y u = \partial_x^+ v^+ = \partial_x^- v^-.$$

We have therefore

$$
\begin{aligned}
(\mathcal{H}^+ f, g) &= 2\int_0^\infty (\partial_y v^+(y), \partial_y u^g(y)) y\mathrm{d}y + 2\int_0^\infty (\partial_x^+ v^+(y), \partial_x^+ u^g(y)) y\mathrm{d}y \\
&= 2\int_0^\infty (-\partial_x^- u^f(y), \partial_y u^g(y)) y\mathrm{d}y + 2\int_0^\infty (\partial_y u^f(y), \partial_x^+ u^g(y)) y\mathrm{d}y \\
(\mathcal{H}^- f, g) &= 2\int_0^\infty (-\partial_x^+ u^f(y), \partial_y u^g(y)) y\mathrm{d}y + 2\int_0^\infty (\partial_y u^f(y), \partial_x^- u^g(y)) y\mathrm{d}y \\
(\mathcal{H} f, g) &= \int_0^\infty (-\partial_x^- u^f(y), \partial_y u^g(y)) y\mathrm{d}y + \int_0^\infty (\partial_y u^f(y), \partial_x^+ u^g(y)) y\mathrm{d}y \\
&\quad + \int_0^\infty (-\partial_x^- u^f(y), \partial_y u^g(y)) y\mathrm{d}y + \int_0^\infty (\partial_y u^f(y), \partial_x^- u^g(y)) y\mathrm{d}y \\
&= \int_0^\infty \left(\begin{pmatrix} -\partial_x^- \\ \partial_y \\ -\partial_x^+ \\ \partial_y \end{pmatrix} u^f(y), \begin{pmatrix} \partial_y \\ \partial_x^+ \\ \partial_y \\ \partial_x^- \end{pmatrix} u^g(y) \right) y\mathrm{d}y \\
&= \int_0^\infty \left(\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_y \\ \partial_x^+ \\ \partial_y \\ \partial_x^- \end{pmatrix} u^f(y), \begin{pmatrix} \partial_y \\ \partial_x^+ \\ \partial_y \\ \partial_x^- \end{pmatrix} u^g(y) \right) y\mathrm{d}y
\end{aligned}
$$

This concludes the proof of Theorems 3 and 4. □

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4. Stochastic representations

Semidiscrete random walks. Let $\mathcal{N}_\sqcup$ be a càdlàg Poisson process with parameter λ. Let $(T_k)_{k\in\mathbb{N}}$ be the instants of jumps. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence of independent Bernouilli variables,

$$\forall k, \quad \mathbb{P}(\varepsilon_k = 1) = \mathbb{P}(\varepsilon_k = -1) = 1/2.$$

This allows us to define the random walk $\mathcal{X}_t \in \mathbb{Z}$ as the compound Poisson process (see Protter [12], Privault [10][11])

$$\mathcal{X}_0 \in \mathbb{Z}, \quad \mathcal{X}_t = \sum_{k=1}^{\mathcal{N}_t} \varepsilon_k$$

Let $\mathcal{Y}_t$ be a standard onedimensional brownian process started at $\mathcal{Y}_0$. We define the semidiscrete random walk $\mathcal{Z}_t$ as $\mathcal{Z}_t := (\mathcal{Y}_t, \mathcal{X}_t) \in \mathbb{R}^+ \times \mathbb{Z}$, i.e.

$$\mathcal{Z}_\prime = (\mathcal{Y}_\prime, \mathcal{X}_\prime), \quad \mathrm{d}\mathcal{Z}_t = (\mathrm{d}B_t, \varepsilon_{\mathcal{N}_t}\mathrm{d}\mathcal{N}_t)$$

Stochastic integrals. Let f defined on $\mathbb{R}^+ \times \mathbb{Z}$ be a smooth function. We have the Itô formula

$$
\begin{aligned}
f(\mathcal{Z}_t) - f(\mathcal{Z}_0) &= \left\{ \int_0^t (\partial_x^0 f)(\mathcal{Z}_{s_-})\mathrm{d}\mathcal{X}_s + \int_0^t \frac{1}{2}(\partial_x^2 f)(\mathcal{Z}_{s_-})\mathrm{d}(\mathcal{N}_s - s) + \int_0^t (\partial_y f)(\mathcal{Z}_{s_-})\mathrm{d}\mathcal{Y}_s \right\} \\
&\quad + \left\{ \frac{1}{2}\int_0^t (\partial_x^2 f + \partial_y^2 f)(\mathcal{Z}_{s_-})\mathrm{d}s \right\}
\end{aligned}
$$

This formula can be derived thanks to telecospic sums involving jump times. We refer to Privault [10][11] for more details.
Quadratic covariation. Let f and g be two semidiscrete harmonic functions in $\mathbb{R}_\star^+ \times \mathbb{Z}$, that is

$$\partial_x^2 f + \partial_y^2 f = \partial_x^2 g + \partial_y^2 g = 0 \quad \text{in} \quad \mathbb{R}_\star^+ \times \mathbb{Z}.$$

We define the corresponding martingales $M_t^f := f(\mathcal{Z}_t)$ and $M_t^g := g(\mathcal{Z}_t)$, so that

$$\begin{aligned}
\mathrm{d}M_t^f &= (\partial_x^0 f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{X}_t + \frac{1}{2}(\partial_x^2 f)(\mathcal{Z}_{t_-})\mathrm{d}(\mathcal{N}_t - t) + (\partial_y f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{Y}_t \\
\mathrm{d}M_t^g &= (\partial_x^0 g)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{X}_t + \frac{1}{2}(\partial_x^2 g)(\mathcal{Z}_{t_-})\mathrm{d}(\mathcal{N}_t - t) + (\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{Y}_t
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{d}[M^f, M^g]_t &= (\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_x^0 g)(\mathcal{Z}_{t_-})\mathrm{d}[\mathcal{X},\mathcal{X}]_t + \frac{1}{4}(\partial_x^2 f)(\mathcal{Z}_{t_-})(\partial_x^2 g)(\mathcal{Z}_{t_-})\mathrm{d}[\mathcal{N},\mathcal{N}]_t \\
&\quad + \left((\partial_x^0 f)(\mathcal{Z}_{t_-})\frac{1}{2}(\partial_x^2 g)(\mathcal{Z}_{t_-}) + (\partial_x^0 g)(\mathcal{Z}_{t_-})\frac{1}{2}(\partial_x^2 f)(\mathcal{Z}_{t_-})\right)\mathrm{d}[\mathcal{X},\mathcal{N}]_t \\
&\quad + (\partial_y f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}[\mathcal{Y},\mathcal{Y}]_t \\
&\quad \left[(\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_x^0 g)(\mathcal{Z}_{t_-}) + \frac{1}{2}(\partial_x^2 f)(\mathcal{Z}_{t_-})\frac{1}{2}(\partial_x^2 g)(\mathcal{Z}_{t_-})\right]\mathrm{d}\mathcal{N}_t \\
&\quad + \left[(\partial_x^0 f)(\mathcal{Z}_{t_-})\frac{1}{2}(\partial_x^2 g)(\mathcal{Z}_{t_-}) + (\partial_x^0 g)(\mathcal{Z}_{t_-})\frac{1}{2}(\partial_x^2 f)(\mathcal{Z}_{t_-})\right]\mathrm{d}\mathcal{X}_t \\
&\quad + (\partial_y f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}t \\
&= \frac{1}{2}[(\partial_x^+ f)(\partial_x^+ g) + (\partial_x^- f)(\partial_x^- g)]\mathrm{d}\mathcal{N}_t \\
&\quad + \frac{1}{2}[(\partial_x^+ f)(\partial_x^+ g) - (\partial_x^- f)(\partial_x^- g)]\mathrm{d}\mathcal{X}_t \\
&\quad + (\partial_y f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}t \\
&= [(\Delta\mathcal{X}_t)_+(\partial_x^+ f)(\partial_x^+ g) + (\Delta\mathcal{X}_t)_-(\partial_x^- f)(\partial_x^- g)]\mathrm{d}\mathcal{N}_t \\
&\quad + (\partial_y f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}t
\end{aligned}$$

where $(\Delta\mathcal{X})_\pm := \max(0, \pm\Delta\mathcal{X})$.

Martingale transform. In order to define a martingale transform N_t^f of M_t^f such that N_t^f allows us to recover the discrete hilbert transform, we recall that the weak formulation (8) of Theorem **??** involves the "martingale transform"

$$\nabla = \begin{pmatrix} \partial_y^- \\ \partial_x^+ \\ \partial_y^+ \\ \partial_x^- \end{pmatrix} \quad \longrightarrow \quad \nabla^\perp = \begin{pmatrix} -\partial_x^- \\ \partial_y^+ \\ -\partial_x^+ \\ \partial_y^- \end{pmatrix}.$$

Let us first rewrite the martingale increments in terms of the $\partial_x^\pm$ derivatives:

$$\begin{aligned}
\mathrm{d}M_t^f &= (\partial_x^0 f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{X}_t + \frac{1}{2}(\partial_x^2 f)(\mathcal{Z}_{t_-})\mathrm{d}(\mathcal{N}_t - t) + (\partial_y f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{Y}_t \\
&= \frac{1}{2}(\partial_x^+ f)(\mathcal{Z}_{t_-})(\mathrm{d}\mathcal{X}_t + \mathrm{d}(\mathcal{N}_t - t)) + \frac{1}{2}(\partial_x^- f)(\mathcal{Z}_{t_-})(\mathrm{d}\mathcal{X}_t - \mathrm{d}(\mathcal{N}_t - t)) \\
&\quad + \frac{1}{2}(\partial_y^- f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{Y}_t + \frac{1}{2}(\partial_y^+ f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{Y}_t.
\end{aligned}$$

The Cauchy-Riemann relations therefore suggest to define

$$\begin{aligned}
\mathrm{d}N_t^f &:= \frac{1}{2}(\partial_y^+ f)(\mathcal{Z}_{t_-})(\mathrm{d}\mathcal{X}_t + \mathrm{d}(\mathcal{N}_t - t)) + \frac{1}{2}(\partial_y^- f)(\mathcal{Z}_{t_-})(\mathrm{d}\mathcal{X}_t - \mathrm{d}(\mathcal{N}_t - t)) \\
&\quad + \frac{1}{2}(-\partial_x^- f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{Y}_t + \frac{1}{2}(-\partial_x^+ f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{Y}_t \\
&= (\partial_y f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{X}_t - (\partial_x^0 f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{Y}_t
\end{aligned}$$

To summarize, let f be harmonic. We have defined N_t^f as the stochastic integral

$$N_t^f := f(\mathcal{Z}_0) + \int_0^t (\partial_y f)(\mathcal{Z}_{s_-})\mathrm{d}\mathcal{X}_s - (\partial_x^0 f)(\mathcal{Z}_{s_-})\mathrm{d}\mathcal{Y}_s.$$

It is now easy to estimate the quadratic covariation

$$\begin{aligned}
\mathrm{d}[M^f, N^f]_t &= (\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y f)(\mathcal{Z}_{t_-})\mathrm{d}[\mathcal{X},\mathcal{X}]_t - (\partial_y f)(\mathcal{Z}_{t_-})(\partial_x^0 f)(\mathcal{Z}_{t_-})\mathrm{d}[\mathcal{Y},\mathcal{Y}]_t \\
&= (\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y f)(\mathcal{Z}_{t_-})\mathrm{d}\mathcal{N}_t^0,
\end{aligned}$$

where we note $\mathcal{N}_t^0 := \mathcal{N}_t - t$ the compensated Poisson process, that is a martingale process.

Notice that M_t^f and N_t^f are not orthogonal martingales with respect to the bracket multiplication $[\cdot,\cdot]$. However, recall that the angular bracket $\langle\cdot,\cdot\rangle$, also known as the conditional quadratic covariation (see Protter [12]), is the compensator of $[\cdot,\cdot]$. But since $\mathcal{N}_t^0$ is a martingale, we have for the angular bracket

$$\mathrm{d}\langle M^f, N^f\rangle_t = 0,$$

that is the martingales M_t^f and N_t^f are orthogonal with respect to the conditional quadratic covariation.

Similarly, the pairing of N_t^f with a test martingale M_t^g leads to the quadratic covariation

$$\begin{aligned}
\mathrm{d}[N^f, M^g]_t &= (\partial_y f)(\mathcal{Z}_{t_-})\left[(\partial_x^0 g)(\mathcal{Z}_{t_-}) + (\Delta_t\mathcal{X})\frac{1}{2}(\partial_x^2 g)(\mathcal{Z}_{t_-})\right]\mathrm{d}\mathcal{N}_t - (\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}t \\
&= (\partial_y f)(\mathcal{Z}_{t_-})[(\Delta_t\mathcal{X})_+(\partial_x^- g)(\mathcal{Z}_{t_-}) + (\Delta_t\mathcal{X})_-(\partial_x^- g)(\mathcal{Z}_{t_-})]\mathrm{d}\mathcal{N}_t \\
&\quad -(\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}t,
\end{aligned}$$

and the conditional quadratic covariation

$$\begin{aligned}
\mathrm{d}\langle N^f, M^g\rangle_t &= (\partial_y f)(\mathcal{Z}_{t_-})\left[(\partial_x^0 g)(\mathcal{Z}_{t_-}) + (\Delta_t\mathcal{X})\frac{1}{2}(\partial_x^2 g)(\mathcal{Z}_{t_-})\right]\mathrm{d}\mathcal{N}_t - (\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}t \\
&= (\partial_y f)(\mathcal{Z}_{t_-})[(\Delta_t\mathcal{X})_+(\partial_x^- g)(\mathcal{Z}_{t_-}) + (\Delta_t\mathcal{X})_-(\partial_x^- g)(\mathcal{Z}_{t_-})]\mathrm{d}\mathcal{N}_t \\
&\quad -(\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}t \\
&= (\partial_y f)(\mathcal{Z}_{t_-})\left[\frac{1}{2}(\partial_x^- g)(\mathcal{Z}_{t_-}) + \frac{1}{2}(\partial_x^- g)(\mathcal{Z}_{t_-})\right]\mathrm{d}t - (\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\mathrm{d}t \\
&= \{(\partial_y f)(\mathcal{Z}_{t_-})(\partial_x^0 f)(\mathcal{Z}_{t_-}) - (\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\}\mathrm{d}t.
\end{aligned}$$

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5. Proof of Theorem 1

It remains to prove the representation formula stated in Theorem 1. Equipped with the martingale representations of the previous sections, it suffices to follow

the lines of Gundy–Varopoulos [5] and Arcozzi [1]. For that, let $(\mathcal{Z}_t)_{-\infty<t\leqslant 0}$ semidiscrete random walks starting at infinity in the upper–half space $\mathbb{Z}\times\mathbb{R}^+$ and stopped at time $t=0$ when reaching the boundary $\mathbb{Z}$. Let f defined on $\mathbb{Z}$, $M_t^f := (P_t f)(\mathcal{Z}_\sqcup)$ the associated martingale, and N_t^f the corresponding martingale transform as defined previously. Finally, introduce the projection operator

$$\mathcal{T}f(x) := \mathbb{E}(N_0^f|\mathcal{Z}_0 = x).$$

It follows that for any test function g defined on $\mathbb{Z}$, and the associated martingale M_t^g, we have

$$\begin{aligned}
(\mathcal{T}f,g)_{\ell^2(\mathbb{Z})} &= \sum_x \mathcal{T}f(x)g(x) = \sum_x \mathbb{E}(N_0^f|\mathcal{Z}_0=x)g(x) = \sum_x \mathbb{E}(N_0^f|\mathcal{Z}_0=x)M_0^g \\
&= \sum_x \mathbb{E}(N_0^f M_0^g|\mathcal{Z}_0=x) = \sum_x \mathbb{E}\left(\int_{-\infty}^0 \mathrm{d}[N^f,M^g]_t \middle\| \mathcal{Z}_0=x\right) \\
&= \sum_x \mathbb{E}\left(\int_{-\infty}^0 \{(\partial_y f)(\mathcal{Z}_{t_-})(\partial_x^0 f)(\mathcal{Z}_{t_-}) - (\partial_x^0 f)(\mathcal{Z}_{t_-})(\partial_y g)(\mathcal{Z}_{t_-})\}\mathrm{d}t \middle\| \mathcal{Z}_0=x\right) \\
&= \sum_x \int_{-\infty}^0 \{(\partial_y f)(y,x)(\partial_x^0 f)(y,x) - (\partial_x^0 f)(y,x)(\partial_y g)(y,x)\}2y\mathrm{d}y \\
&= (\mathcal{H}f,g)
\end{aligned}$$

where we used the fact that M_0^g depends only on $\mathcal{Z}_0$ but not on the trajectory, where we used the formula of the previous section for the quadratic covariations, and finally the fact that the density of the background noise $\mathcal{Z}_t$ is equal to $2y$.

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